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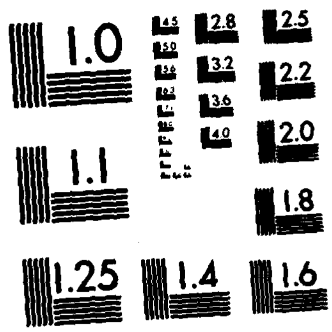

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ON THE FATOU INEQUALITY

by

Aryeh Dvoretzky
Hebrew University

TECHNICAL REPORT NO. 23
OCTOBER 1983

PREPARED UNDER CONTRACT
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<u>On The Fatou Inequality</u>

Aryeh Dvoretzky

1. Introduction

The classical Fatou inequality ~~[6]~~

$$(1.1)\qquad \int \underline{\lim}\, X_n(\omega)d\omega \le \underline{\lim} \int X_n(\omega)d\omega \ ,$$

for non-negative measurable functions X_n is intimately connected with problems of convergence of random variables (r.v.).

The present paper focuses on the study of a modified form of ~~(1.1)~~ the Fatou inequality which has important applications in the theory of convergence of r.v.

We formulate our results in the language of probability. $(\Omega,\mathfrak{F},P)$ is a probability space. $(\mathfrak{F}_n)_{n\in N}$ is a non-decreasing sequence of sub-σ-algebras of $\mathfrak{F}$. The sequences $(X_n)_{n\in N}$ of r.v. are adapted to $(\mathfrak{F}_n)$, i.e., X_n is $\mathfrak{F}_n$-measurable for all n .

The stopping times associated with $(\mathfrak{F}_n)$ are the mappings $t\colon \Omega \to N$ such that $\{t=n\} = \{\omega; t(\omega) = n\} \in \mathfrak{F}_n$ for all n . We put $\mathfrak{F}_t = \{A; A \cap \{t=n\} \in \mathfrak{F}_n \text{ for all } n\}$. X_t is the r.v. $(X_t)(\omega) = X_{t(\omega)}(\omega)$, it is $\mathfrak{F}_t$-measurable. A bounded stopping time (b.s.t.) is a stopping time assuming only finitely many values. We donote by T the family of all bounded stopping times.

We use the letter E to denote expectation. The variant of (1.1) studied in this paper is

$$(1.2)\qquad E\,\underline{\lim}|X_n| \le \underline{\lim}_{t\in T} E|X_t| \ .$$

(Throughout the paper lim, $\underline{\lim}$, etc. refer to the relevant index increasing to infinity).

It is easy but not very interesting to establish the conditions for equality in (1.1). On the other hand, W. D. Sudderth [7] showed that if the stopping times in (1.2) are not restricted to be bounded, then the inequality reduced to an equality. (The proof in [7] relies on a martingale convergence theorem; the result can also be easily obtained from Lemma 4.1 of the present paper).

It is, however, the bounded stopping times that play a major part in various generalizations of the theory of martingales. The pioneering work in this connection is due to J. R. Baxter [2] and D. G. Austin, G. A. Edgar and A. Ionescu Tulcea (= A. Bellow) [1].

<u>All stopping times considered throughout the paper are bounded</u>. A detailed study of the excess of the right side of (1.2) over the left side is the core of the present paper.

<u>The Fatou discrepancy</u> is the set function given by

$$\underline{\lim}\, E|X_t|1_A - E\,\underline{\lim}|X_n|1_A \quad , \tag{1.3}$$

where 1_A (as always in the paper) denotes the indicator of the set A . It plays a central role in the present study.

For simplicity of statements we make the following assumptions:

<u>A1</u>. $\mathcal{F} = \sigma(\mathcal{A})$, <u>i.e.</u>, the σ-<u>algebra</u> $\mathcal{F}$ <u>is generated by the algebra</u> $\mathcal{A}$ where $\mathcal{A} = \bigcup_{n\in N} \mathcal{F}_n$.

<u>A2</u>. $\underline{\lim}|X_n|$ <u>is almost surely (a.s.) finite, i.e.</u>, $P(\underline{\lim}|X_n| < \infty) = 1$.

The second assumption is equivalent to the right side of (1.2) being σ-finite over $\mathcal{A}$, i.e., to the existence of sets $A \in \mathcal{A}$ with

$P(A)$ arbitrarily close to 1 for which $E\ \underline{\lim}\ E|X_t|1_A < \infty$. That this condition implies A2 follows from (1.2), the implication in the other direction is by a standard argument in the theory of stopping times (used also in the proof of Lemma 3.2).

In section 2, we consider the set function $\mu^*(A) = \underline{\lim}\ E|X_t|1_A$, $(A \in \mathcal{G})$, show that it is finitely additive and introduce through μ^* a measure μ on $\mathfrak{F}$. In the next section we define, for $A \in \mathcal{G}$, a set function $\phi(A) = \mu^*(A) - \mu_0(A)$, where μ_0 is the absolutely continuous part of μ relative to P, and show (Theorem 3.1) that it is the Fatou discrepancy (1.3). The fact that ϕ dwells on small sets (Lemma 3.1) is fundamental.

Whereas in sections 2 and 3 we consider only the absolute values $|X_n|$ of the r.v. X_n, we turn in 4 to the r.v. themselves and their possible limits. We denote by $\overline{C}$ the set of r.v. which are pointwise limits of X_n and by C the subset of integrable r.v. in $\overline{C}$. The main result here is (Theorem 4.1) $\underline{\lim}\ E|X_t - Y| = \phi(\Omega) + \rho(Y,C)$ for all integrable r.v. Y where $\rho(Y,C)$ is the L_1 distance between Y and C. This yields (Corollary 4.1) a characterization of C (quite different from that given by A. Bellow [4] in terms of sub-martingales). We draw attention to a useful approximation result (Lemma 4.1).

In section 5 we consider simultaneous approximations. The principal result here (Theorem 5.1) is that we can associate with every $Y \in \overline{C}$ a sequence of bounded stopping times $t_n(Y)$ such that $X_{t_n(Y)} \to Y$ a.s. and, moreover, if $Y - Z$ is integrable then $X_{t_n(Y)} - X_{t_n(Z)} \to Y - Z$ in L_1 norm. Results of this nature were proved in [5], [3] and [4] (indeed they are the key to the proof of the amart convergence theorem). The main point here is not in weakening the assumptions and extending the conclusion but in having the same sequence $t_n(Y)$ for all Z.

Section 6 introduces a finitely additive real-valued set function $\tilde{\phi}$ on $\mathcal{G}$ having the property (Theorem 6.1) that there is associated with every $Y \in C$ a sequence of bounded stopping times $t_n(Y)$ such that, if $\underline{\lim}\, E|X_t| < \infty$, we have $E\, X_{t_n(Y)} V \to EYV + \sum_{i=1}^{m} \lambda_i \tilde{\phi}(A_i)$ for every simple r.v. $V = \sum_{i=1}^{m} \lambda_i 1_{A_i}$ $(A_i \in \mathcal{G})$. In particular, $E(X_{t_i(Y)} - X_{t_j(Z)})\, V \to E(Y - Z)V$ for all $Y, Z \in Z$ when $i,j \to \infty$ independently of one another (unlike the situation in the preceding paragraph).

Except in the last section, only real-valued r.v. are considered. In 7 the results are extended to r.v. assuming values in a Banach space. Those of 6 carry through to finite dimensional Banach spaces; all other results remain valid (with minor variations) in infinitely dimensional Banach spaces.

We use the letters t, s, τ to denote bounded stopping times and X,Y,Z,V to represent random variables.

2. The set function μ^* and the measure μ.

Definition 2.1. *We define a set function $\mu^*(A)$ on $\mathcal{G}$ by*

$$\mu^*(A) = \underline{\lim}\, E|X_t| 1_A, \quad (A \in \mathcal{G}). \tag{2.1}$$

We recall that (2.1) is equivalent to the following statement. There exists a sequence $(t_n)_{n \in \mathbb{N}}$ of bounded stopping times increasing to infinity such that

$$(2.2) \qquad \lim E|X_{t_n}| 1_A = \mu^*(A) \quad ,$$

while for all sequences $(s_n)_{n \in \mathbb{N}}$ of b.s.t. with $s_n \to \infty$ we have $\overline{\lim}\ E|X_{s_n}| 1_A \geq \mu^*(A)$. μ^* is a monotone set function from $\mathcal{G}$ to $[o,\infty]$.

When it is necessary to exhibit the dependence of μ^* on the sequence X_n we shall write $\mu^*(\cdot,(X_n))$. Similarly for $\mu,\ldots,\phi$ which will be introduced later.

Lemma 2.1. μ^* is finitely additive on $\mathcal{G}$, i.e.,

$$(2.3) \qquad \mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

for disjoint $A,B \in \mathcal{G}$.

Proof. Let $t_n \to \infty$ be b.s.t. for which (2.2) holds with A replaced by $A \cup B$. Then

$$(2.4) \qquad \mu^*(A \cup B) \geq \underline{\lim}\ E|X_{t_n}| 1_A + \underline{\lim}\ E|X_{t_n}| 1_B$$
$$\geq \underline{\lim}\ E|X_t| 1_A + \underline{\lim}\ E|X_t| 1_B = \mu^*(A) + \mu^*(B) \ .$$

On the other hand, let $A \in \mathfrak{F}_k$ and $t_n \to \infty$ satisfy (2.2) and $s_n \to \infty$ be b.s.t. satisfying $E|X_{s_n}| 1_B \to \mu^*(B)$. Assume further $s_n \geq k$, $t_n \geq k$ for all n . Putting $\tau_n = t_n$ for $\omega \in A$ and $\tau_n = s_n$ for $\omega \notin A$

we have

$$(2.5) \qquad \mu^*(A \cup B) \le \lim E|X_{\tau_n}|1_{A\cup B} = \lim E|X_{t_n}|1_A + \lim E|X_{s_n}|1_B$$
$$= \mu^*(A) + \mu^*(B) \ .$$

Together with (2.4) this yields (2.3). □

The following easy result will be useful.

<u>Lemma 2.2</u>. <u>If</u> $\mu^*(A) < \infty$ <u>and</u> $t_n \to \infty$ <u>are bounded stopping times satisfying (2.2) then</u>

$$(2.6) \qquad \lim E|X_{t_n}|1_B = \mu^*(B) \ ,$$

<u>holds for every</u> $B \in \mathcal{G}$ <u>with</u> $B \subset A$.

<u>Proof</u>. Otherwise, since $\mu^*(B) \le \mu^*(A) < \infty$, we would have $\overline{\lim}\, E|X_{t_n}|1_B > \mu^*(B)$. Taking $C = A \setminus B$ we have, by definition, $\underline{\lim}\, E|X_{t_n}|1_C \ge \mu^*(C)$. Combining the two relations we obtain $\overline{\lim}\, E|X_{t_n}|1_A > \mu^*(B) + \mu^*(C) = \mu^*(A)$ contradicting (2.6). □

μ^* may not be σ - additive on $\mathcal{G}$ (see Remark 2.1). We can, however, derive in the standard way a σ - additive function.

For every $A \in \mathcal{G}$ we put

$$(2.7) \qquad \mu(A) = \inf\left\{ \sum_{n\in N} \mu^*(A_n);\ A \subset \bigcup_{n\in N} A_n \ , A_n \in \mathcal{G} \ , n \in N \right\} .$$

Since μ^* is monotone we may confine (2.7) to countable partitions of A into sets $A_n \in \mathcal{G}$. μ^* is a monotone function from $\mathcal{G}$ to $[o,\infty]$. Clearly

$$\mu(A) \le \mu^*(A) , \quad (A \in \mathcal{G}). \tag{2.8}$$

Lemma 2.3. μ is countably additive on $\mathcal{G}$.

Proof. Let $A,B \in \mathcal{G}$ be disjoint and $C = A \cup B$. Since each partition $(C_n)_{n \in N}$ of C corresponds to the partitions $(A \cap C_n)$ and $(B \cap C_n)$ of A and B and vice versa the finite additivity of μ follows at once from that of μ^*.

We have to show that if $A_n \in \mathcal{G} (n \in N)$ are disjoint and $A = \bigcup A_n \in \mathcal{G}$ then

$$\mu(\bigcup A_n) = \sum \mu(A_n) . \tag{2.9}$$

Since μ is finitely additive and monotone the left side of (2.9) is $\ge$ the right side. Thus it remains to show that $\mu(\bigcup A_n) \le \sum \mu(A_n)$ and this only when the right side is finite. Let $\varepsilon > 0$ be given and let, for every n, $A_{n,j} (j \in N)$ be a partition of A_n into sets of $\mathcal{G}$ satisfying $\sum_{j \in N} \mu^*(A_{n,j}) < \mu(A_n) + \varepsilon/2^n$. Then $A_{n,j}, (j,n \in N)$, constitute a countable partition of A and $\sum_{j,n} \mu^*(A_{n,j}) < \sum_n \mu(A_n) + \varepsilon$. □

Since, by (2.8) and Assumption A2, μ is σ-finite on $\mathcal{G}$ it extends uniquely to a measure on $\mathfrak{F} = \sigma(\mathcal{G})$. We denote this measure by the same letter μ.

Definition 2.2. μ is the measure induced on $\mathfrak{F}$ by (2.7).

Remarks. 2.1. μ^* need not be countably additive on $\mathcal{G}$. Thus for $\Omega = (0,1]$, P the lebesgue measure, $X_n = 2^n 1_{(0,2^{-n}]}$ and $\mathfrak{F}_n = \sigma(X_1,\ldots,X_n)$, we have $\mu^*(A_n) = 0$ for all n but $\mu^*(\Omega) = \mu^*(\cup A_n) = 1$, where $A_n = (2^{-n}, 2^{-n+1}]$.

2.2. As seen from the proof of Lemma 2.1, $\underline{\lim}\, E|X_t| 1_{A\cup B} \geq \underline{\lim}\, E|X_t| 1_A + \underline{\lim}\, E|X_t| 1_B$ for all disjoint measurable A and B (not necessarily in $\mathcal{G}$). Moreover, equality holds if either $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

If, however, neither A nor B are in $\mathcal{G}$ there may occur a sharp inequality. Indeed, taking in the preceding example $A = \cup(2^{-(2n)!}, 2^{-(2n-1)!}]$ and B as its complement, we have $\underline{\lim}\, E\, X_t 1_A = \underline{\lim}\, E\, X_t 1_B = 0$ but $\mu^*(\Omega) = \underline{\lim}\, E\, X_t = 1$.

2.3. For measurable $A \notin \mathcal{G}$ we need not have $\mu(A) \leq \underline{\lim}\, E|X_t| 1_A$ as in (2.8). Indeed, modify the example in Remark 2.1 through replacing $\Omega = (0,1]$ by $\Omega = [0,1]$. Then $\mu(\{0\}) = 1$.

3. The set functions ϕ, Δ and the measures μ_0, μ_1.

Since μ is σ-finite it can be decomposed according to Lebesgue.

Definition 3.1. μ_0 and μ_1 are, respectively, the absolutely continuous and singular components of the measure μ relative to P.

μ_0 and μ_1 are, of course, defined for all measurable sets. The following definition introduces a set function on $\mathcal{G}$ which occupies a central position in the present paper.

Definition 3.2. The Fatou discrepancy of the sequence of random variables (X_n) relative to the σ-fields $(\mathfrak{F}_n)$ is given by

$$\phi(A) = \mu^*(A) - \mu_0(A) , \qquad (A \in \mathcal{G}) , \tag{3.1}$$

where, in case $\mu_0(A) = \infty$, *this is to be interpreted as* $\sup\{\phi(B); B \subset A, B \in \mathcal{G}, \mu_0(B) < \infty\}$.

We put

$$\Delta(A) = \mu^*(A) - \mu(A) , \qquad (A \in \mathcal{G}) , \tag{3.2}$$

where again, if $\mu(A) = \infty$, this is to be interpreted as above. Clearly

$$\phi = \Delta + \mu_1 . \tag{3.3}$$

Δ and ϕ are finitely additive (on their domain of definition $\mathcal{G}$) . One could loosely describe Δ and μ_1 as the dissipative and singular components, respectively, of μ^* .

It follows from Theorem 3.1 that μ_0 depends only on the sequence (X_n) and not on the filtration $(\mathfrak{F}_n)$. It then follows from (3.1) that any two filtrations which yield the same value of $\mu^*(A)$ will give also the same value of $\phi(A)$. Remark 3.1. will point out that this is not true for $\Delta(A)$ or $\mu_1(A)$. This is the reason why it is not Δ or μ_1 but their sum (3.3) that figures prominently in our results.

Theorem 3.1. *We have for all measurable sets*

$$\mu_0(A) = E \,\underline{\lim}\, |X_n| 1_A , \qquad (A \in \mathfrak{F}) . \tag{3.4}$$

The following result is the key to the proof of Theorem 3.1.

Lemma 3.1. ϕ *dwells on small sets, i.e., for every* $\varepsilon > 0$ *there exists* $A \in \mathcal{G}$ *satisfying*

$$P(A) > 1-\varepsilon\ , \qquad \phi(A) < \varepsilon\ . \tag{3.5}$$

Proof. (This result is equivalent to the apparently more general statement: given $B \in \mathcal{G}$ and $\varepsilon > 0$ there exists a set $A \subset B$, $A \in \mathcal{B}$ such that $P(A) > P(B) - \varepsilon$ and $\phi(A) < \varepsilon$.) Since by Assumption A2, Ω contains sets of $\mathcal{G}$ with probability arbitrarily close to 1 with finite μ^* it is enough to prove (3.5) for the case $\mu^*(\Omega) < \infty$. Let $A_n (n \in N)$ be a partition of Ω into sets of $\mathcal{G}$ for which $\Sigma\mu^*(A_n) < \mu(\Omega) + \varepsilon/2$. Then $\Delta(\overset{m}{\underset{1}{\cup}} A_n) = \mu^*(\overset{m}{\underset{1}{\cup}} A_n) - \mu(\overset{m}{\underset{1}{\cup}} A_n) < \varepsilon/2$ for all $m \in N$ and $P(\overset{m}{\underset{1}{\cup}} A_n) > 1-\varepsilon/2$ for large m. Hence there exists $A' \in \mathcal{G}$ with $P(A') > 1 - \varepsilon/2$ and $\Delta(A') < \varepsilon/2$. Similarly, since $\mu_1 \perp P$ and $\mathcal{F} = \sigma(\mathcal{G})$, there exists $A'' \in \mathcal{G}$ with $P(A'') > 1 - \varepsilon/2$ and $\mu_1(A'') < \varepsilon/2$. $A = A' \cap A''$ satisfies (3.5). $\square$

Lemma 3.2. *For every* $\varepsilon > 0$ *and* $k \in N$ *there exists* $A \in \mathcal{G}$ *and a bounded stopping time* $t > k$ *such that*

$$P(A) > 1 - \varepsilon\ , \qquad E|X_t|1_A < E\ \underline{\lim}|X_n|1_A + \varepsilon\ . \tag{3.6}$$

Proof. Let A be the set described in the preceding Lemma with ε replaced by $\varepsilon/4$. We may assume $\mu^*(A) < \infty$. Since μ_0 is absolutely continuous and $\mu_0(A) < \infty$ there exists $\delta > 0$ such that $\mu_0(B) < \varepsilon/4$

for every $B \subset A$ with $P(B) < \delta$. Then also $\mu^*(B) = \mu_0(B) + \phi(B) < \varepsilon/2$.

Let Y be a simple (i.e., assuming only finitely many values) $\mathcal{G}$-measurable r.v. with $P(\underline{\lim}\, X_n < Y < \underline{\lim}\, X_n + \varepsilon/4) > 1 - \delta$. Y is $\mathcal{F}_m$-measurable for some m and we may take $m > k$. Let the b.s.t. $s > m$ be such that $P(|X_s| < Y) > 1 - \delta$. Let $C = \{|X_s| < Y\}$ and D be its complement. Let the b.s.t. $\tau > s$ be such that $E|X_\tau| 1_{A \cap D} < \mu^*(A \cap D) + \varepsilon/4 < 3\varepsilon/4$.
Put $t = s\, 1_C + \tau\, 1_D$, then

$$E|X_t| 1_A = E|X_t| 1_{A\cap C} + E|X_t| 1_{A\cap D} < E|Y| 1_{A\cap C} + 3\varepsilon/4$$
$$< E\, \underline{\lim}|X_n| 1_A + \varepsilon . \qquad \square$$

<u>Proof of Theorem 3.1</u>. As remarked in the introduction $\underline{\lim}|X_n| 1_A$ is finite a.s. and both sides of (3.4) are σ-finite measures on $\mathcal{F} = \sigma(\mathcal{G})$. *It suffices therefore to prove the assertion for* $A \in \mathcal{G}$ with $\mu^*(A) < \infty$.

Without loss of generality we may take $A = \Omega$ and assume $\mu_0(\Omega) < \infty$, $E\, \underline{\lim}|X_n| < \infty$.

Let $\varepsilon > 0$ be given, choose $A \in \mathcal{G}$ satisfying (3.5) and denote its complement by B. Then $E\, \underline{\lim}|X_n| = E\, \underline{\lim}\, |X_n| 1_A + E\, \underline{\lim}|X_n| 1_B \le \mu^*(A) + E\, \underline{\lim}|X_n| 1_B = \mu_0(A) + \phi(A) + E\, \underline{\lim}|X_n| 1_B \le \mu_0(\Omega) + \varepsilon + E\, \underline{\lim}|X_n| 1_B$. Since $\underline{\lim}|X_n|$ is integrable letting $\varepsilon \to 0$ we obtain

$$(3.6) \qquad E\, \underline{\lim}|X_n| \le \mu_0(\Omega) .$$

For the set A of Lemma 3.2 we have $\mu^*(A) \le E\, \underline{\lim}|X_n| + \varepsilon$. Denoting by B the complement of A we have $\mu_0(\Omega) = \mu_0(A) + \mu_0(B) \le E\, \underline{\lim}|X_n| + \varepsilon + \mu_0(B)$. Letting $\varepsilon \to 0$ we obtain the reverse of inequality (3.6). $\square$

Theorem 3.2. The sign of equality holds in the Fatou inequality (1.2)

$$E\ \underline{\lim}|X_n| \le \underline{\lim}\ E|X_t|\ ,$$

if and only if $\phi(\Omega) = 0$.

The following result will be useful in the next section.

Lemma 3.3. Let $(\overline{X}_n)$, $n \in N$ be a sequence of random variables adapted to $(\mathfrak{F}_n)$ and put $\overline{\mu}^*(\cdot) = \mu^*(\cdot,(\overline{X}_n)), \phi(\cdot) = \overline{\phi}(\cdot,(\overline{X}_n))$ etc. If $\sup_{n\in N}||\overline{X}_n| - |X_n||$ is integrable then $\overline{\phi} = \phi$.

Proof. Putting $Y = \sup||\overline{X}_n| - |X_n||$ we note that the set function $\nu = \overline{\mu}^* - \mu^*$ satisfies $|\nu(A)| \le EY1_A$ for $A \in \mathcal{G}$. Since ν is finitely additive it follows that it is countably additive and hence (extends to) a signed absolutely continous measure on $\mathfrak{F}$. From Definition 2.1 it follows at once that $\overline{\mu} = \mu + \nu$, thus $\overline{\Delta} = \Delta$. Since ν is absolutely continuous the singular parts of μ and $\overline{\mu}$ are the same. Hence $\overline{\mu}_1 = \mu_1$. □

Remarks 3.1. In the example considered in Remark 2.3 we have $\mu_1(\Omega) = 1$, $\Delta(\Omega) = 0$. If we replace $\mathfrak{F}_n = \sigma(X_1,\ldots,X_n)$ by $\mathfrak{F}_n = \sigma(\{0\},X_1,\ldots,X_n)$ we will have $\mu_1(\Omega) = 0$, $\Delta(\Omega) = 1$.

3.2. Let $\Omega = [0,1]$, P be the lebesgue measure and $\mathfrak{F}_n$ be the algebra generated by $((i-1)/2^n, i/2^n)$, $i = 1,\ldots,2^n$ and let $X_n = 2^n \sum_{i=1}^{2^n} 1_{A_{i,n}}$ where $A_{i,n} = (i/2^n - 1/2^{2n}, i/2^n)$. Then $X_n \to 0$ a.s. but μ and μ_0 coincide with the Lebesgue measure. Here, of course, X_n is not adapted to $\mathfrak{F}_n$, thus exhibiting the necessity of this requirement for the validity of our theorems.

3.3. It follows from Theorem 3.1 and Lemma 2.2 that if $\mu^*(\Omega) < \infty$ and the b.s.t. $t_n \to \infty$ satisfy $E|X_{t_n}| \to \mu^*(\Omega)$ then $E|X_{t_n}|1_A \to E\,\underline{\lim}|X_n|1_A + \phi(A)$ for every $A \in \mathcal{G}$. Hence, even when $\mu^*(\Omega) = \infty$, there exists a sequence of b.s.t. $\tau_n \to \infty$ such that $E|X_{\tau_n}|1_A \to E\,\underline{\lim}|X_n|1_A + \phi(A)$ holds for every $A \in \mathcal{G}$.

4. The cluster set.

Until now, we have considered only the absolute value $|X_n|$ of the r.v. X_n. From now on we shall be concerned with the r.v. themselves.

Definition 4.1. The cluster set $\overline{C} = \overline{C}(X_n)$ of the sequence of random variables $(X_n)_{n\in N}$ is the set of random variables X which are a.s. pointwise limits of the random variables X_n, i.e., which satisfy

$$(4.1) \qquad P(\underline{\lim}|X_n - X| = 0) = 1 .$$

C is the subset of $\overline{C}$ consisting of the integrable random variables in the cluster set, i.e., $C = \overline{C} \cap L_1(\Omega,\mathfrak{F},P)$.

C is a closed set in the metric space $L_1 = L_1(\Omega,\mathfrak{F},P)$. The L_1 distance from an integrable r.v. Y to C is given by $\rho(Y,C) = \inf\{E|Y-X|;\ X \in C\}$. Unless C is empty there exists $X' \in C$ for which the infimum is achieved.

If $Y \in \overline{C}(X_n)$ then, obviously, $|Y| \in C(|X_n|)$. Conversely, if $Y \in \overline{C}(|X_n|)$ there exists $Z \in C(X_n)$ with $|Z| = Y$. Indeed, there exist b.s.t. $t_n \to \infty$ for which $|X_{t_n}| \to Y$ (this is a well-known result, see e.g. [1]; it also follows from Lemma 4.1). We may take $Z = Y$ on $\{\overline{\lim}\, X_{t_n} > 0\}$ and $Z = -Y$ otherwise.

It follows from this observation that if $X' \in C$ and $E|X' - Y| = \rho(Y,C)$ then $P(\underline{\lim}|X_n - Y| = |X' - Y|) = 1$.

$\overline{C}(X_n)$ is not empty since $\lim|X_n| \in \overline{C}(|X_n|)$. Also, C is not empty if and only if

$$\text{(4.2)} \qquad E\ \underline{\lim}|X_n| < \infty .$$

A special case of the following approximation result was used in the proof of Theorem 3.1.

Lemma 4.1. *For every* $\varepsilon > 0$ *there exists a set* $A \in \mathcal{G}$ *with* $P(A) > 1-\varepsilon$ *such that for every* $Y \in \overline{C}$ *there exist arbitrarily large bounding stopping times* t *satisfying*

$$\text{(4.3)} \qquad E|X_t - Y|1_A < \varepsilon .$$

If $Y \in C$ *there exist sequences of* $\mathcal{G}$-*measurable set* A_n *and bounded stopping times* $t_n \to \infty$ *such that* $X_{t_n} 1_{A_n} \to Y$ *in* L_1-*norm*.

Proof. The first part is a restatement of Lemma 3.2 for the sequence $(X_n - Y)$. The second part follows upon denoting by t_n and A_n a b.s.t. and set satisfying (4.3) with $\varepsilon = 1/n$.

Theorem 4.1. *If* Y *is an integrable random variable then*

$$\text{(4.4)} \qquad \underline{\lim}\ E|X_t - Y| = \phi(\Omega) + \rho(Y,C) .$$

Proof. By Lemma 3.3 the ϕ corresponding to the sequence of r.v. $(X_n - Y)$ is the same as that corresponding to the sequence (X_n).

Assume first $C \neq \emptyset$, then we have by Theorem 3.1 $\underline{\lim}\, E|X_t - Y| = \phi(\Omega) + E\,\underline{\lim}|X_n - Y|$ and, as remarked above, $\rho(Y,C)$ equals the second summand on the right.

It remains to check that if (4.2) fails then $\underline{\lim}\, E|X_t - Y| \geq \underline{\lim}\, E|X_t| - E|Y| = \infty$. □

Corollary 4.1. *If* Y *is an integrable random variable then*

$$\underline{\lim}\, E|X_t - Y| \geq \phi(\Omega) , \tag{4.5}$$

and if

$$\underline{\lim}\, E|X_t - Y| < \infty , \tag{4.6}$$

equality occurs in (4.5) when and only when $Y \in C$.

The following is an immediate extension of Theorem 4.1. (We recall that Assumption *A 2* implies $\bar{C} \neq \emptyset$).

Corollary 4.2. *Let* $Z \in \bar{C}$ *and denote by* C_Z *the set of integrable random variables* X *for which* $Z + X \in \bar{C}$. *Then, for every integrable random variable* Y *we have*

$$\underline{\lim}\, E|X_{t_n} - (Z + Y)| = \phi(\Omega, (X_n - Z)) + \rho(Y, C_Z) . \tag{4.7}$$

The following result is useful.

Lemma 4.2. If $\phi(\Omega) < \infty$, Y is an integrable random variable and the bounded stopping times $t_n \to \infty$ satisfy

$$\lim E|X_{t_n} - Y| = \phi(\Omega)$$

then $X_{t_n} \to Y$ in probability.

Proof. If the conclusion fails there exist $\delta > 0$ and a subsequence (s_n) of (t_n) such that $E|X_{s_n} - Y|1_B > \delta$ for all B with $P(B) > 1-\delta$ and all n.

Take $\varepsilon < \delta$, let A be the set described in Lemma 4.1 and let B be its complement. Then $\underline{\lim}\, E|X_{s_n} - Y| \geq \underline{\lim}\, E|X_{s_n} - Y|1_A + \underline{\lim}\, E|X_{s_n} - Y|1_B \geq \delta + \phi(B) \geq \phi(\Omega) + \delta - \varepsilon > \phi(\Omega)$. □

Remarks 4.1. Lemma 4.1 implies that every $Y \in \overline{C}$ is the limit in probability of a sequence X_{t_n} with $t_n \to \infty$. (This actually characterizes $\overline{C}$ since a subsequence of X_{t_n} will converge a.s. to Y.) This result about approximation in probability is well known and has been extensively used in the study of Amarts and related topics. It is explicitly stated and proved in [1] and is implicit and crucial in [2].

4.2. Notice that the set A in the approximation Lemma 4.1 does not depend on Y.

4.3. If $Z, Z' \in \overline{C}$ and $E|Z' - Z| < \infty$ then, with the notation of Corollary 4.2, $C_{Z'} = C_Z$ thus the right side of (4.7) does not change if Z is replaced by Z'.

5. Simultaneous approximations.

If $Y, Z \in C$ and $\phi(\Omega) = 0$ there exist sequences (s_n), (t_n) of b.s.t. such that $X_{s_n} \to Y$, $X_{t_n} \to Z$ in L_1 norm and hence also $X_{s_i} - X_{t_j} \to Y - Z$ in L_1 norm as $i,j \to \infty$ independently of one another.

The situation is quite different when $\phi(\Omega) > 0$. In the example of Remark 2.1, where C consists of the one r.v. $Y \equiv 0$, if t and s are any b.s.t. with $\min t > \max s$ we have $E|X_t - X_s| > 1$. Thus there cannot exist sequences (s_n), (t_n) satisfying $E|X_{s_i} - X_{t_j}| \to E|Y - Y| = 0$ as $i,j \to \infty$ independently of one another. If, however, we let $j = j(i)$ then the L_1 approximation of $Y - Z$ can be achieved.

Theorem 5.1. *To every $Y \in \overline{C}$ there exists an increasing sequence of bounded stopping times $t_n(Y)$, $n \in N$, satisfying $X_{t_n(Y)} \to Y$ a.s. and such that for any Y, Z with $E|Y - Z| < \infty$ we have*

$$X_{t_n(Y)} - X_{t_n(Z)} \to Y - Z, \tag{5.1}$$

in L_1-norm.

Proof. Let A_n be a set having the properties described in Lemma 4.1 for $\epsilon = 1/n^2$. Let k_n be such that $A_n \in \mathfrak{F}_{k_n}$ and $s_n \geq k_n$ be a b.s.t. satisfying (4.3) for these A_n and ϵ. We may take $s_n > s_{n-1}$. Let (τ_n) be any increasing sequence of b.s.t. satisfying $\tau_n \geq k_n$. Put $t_n(Y) = s_n(Y)1_{A_n} + \tau_n 1_{B_n}$ where B_n is the complement of A_n. Then, for any $Y, Z \in \overline{C}$ we have $E|X_{t_n(Y)} - X_{t_n(Z)} - (Y - Z)| = E|X_{t_n(Y)} - Y - (X_{t_n(Z)} - Z)|1_{A_n} + E|Y - Z|1_{B_n} < 2/n^2 + E|Y - Z|1_{B_n} \to 0$ since $Y - Z$ is integrable. □

A similar argument yields,

Theorem 5.2. *If C is not empty then for every integrable Y there exists an increasing sequence of bounded stopping times $t_n(Y)$ such that $X_{t_n(Y)} \to Y'$ a.s. where $Y' \in C$ is nearest (in L_1-metric) to Y, i.e., $E|Y - Y'| = \rho(Y,C)$, and*

$$\lim E|X_{t_n(Y)} - X_{t_n(Z)} - (Y - Z)| \le \rho(Y,C) + \rho(Z,C) , \tag{5.2}$$

for all integrable random variables Y,Z.

The assertion implies that the limit in (5.2) exists.

Remarks 5.1. J. R. Baxter [3], following an extension of the Fatou inequality by R. V. Chacon [5], proved that if $\underline{\lim}\, E|X_n| < \infty$ and $Y,Z \in C$ there exist b.s.t. s_n, $t_n \to \infty$ satisfying $X_{s_n} - X_{t_n} \to Y - Z$ in L_1-norm, these sequences were, however, dependent on the pair Y,Z. (See also A. Bellow [4]).

5.2. If $Y - Z$ is not integrable then, by Fatou, $E|X_{t_n(Y)} - X_{t_n(Z)}| \to \infty$.

5.3. Theorems 5.1 and 5.2 can be extended to the situation considered in Corollary 4.2.

6. A signed set function $\tilde{\phi}$.

For simplicity of statements we assume in this section

$$\phi(\Omega) < \infty , \tag{6.1}$$

and put $\underline{X} = \underline{\lim}\, X_n$.

Definition 6.1. Let, for $A \in \mathcal{G}$,

$$(6.2) \qquad \phi^+(A) = \sup\{\overline{\lim}\, E(X_{t_n} - \underline{X})^+ 1_A \; ; \; \lim E|X_{t_n} - \underline{X}| 1_A = \phi(A)\} ,$$

(i.e., we consider only such sequences of bounded stopping times $t_n \to \infty$ for which the condition holds). Let $\phi^-(A) = \phi(A) - \phi^+(A)$ and $\tilde{\phi}(A) = \phi^+(A) - \phi^-(A)$.

Clearly, $0 \le \phi^+ \le \phi$ and $|\tilde{\phi}| \le \phi$.

Lemma 6.1. ϕ^+, ϕ^- and $\tilde{\phi}$ are finitely additive on $\mathcal{G}$.

Proof. It suffices to prove this for ϕ^+ . We remark that the sup in (6.2) is achieved. Let A, B be disjoint with $A, B \in \mathcal{F}_k$. Let the increasing sequences of b.s.t. (s_n) and (t_n) satisfy $s_1, t_1 > k$ and $E(X_{t_n} - \underline{X})^+ 1_A \to \phi^+(A)$, $E|X_{t_n} - \underline{X}| 1_A \to \phi(A)$ and similarly for s_n and B . Let $\tau_n = t_n$ on A and $= s_n$ on its complement. Then $E|X_{\tau_n} - \underline{X}| 1_{A \cup B} \to \phi(A) + \phi(B) = \phi(A \cup B)$ while $\phi^+(A \cup B) \ge \underline{\lim}\, E|X_{\tau_n} - \underline{X}| 1_{A \cup B} = \phi^+(A) + \phi^+(B)$. Starting with b.s.t. τ_n satisfying $E(X_{\tau_n} - \underline{X})^+ 1_{A \cup B} \to \phi^+(A \cup B)$ and $E|X_{\tau_n} - \underline{X}| 1_{A \cup B} \to \phi(A \cup B)$ we obtain (see Remark 3.3) the opposite inequality. $\square$

We need the following strengthened and extended version of Lemma 2.2.

Lemma 6.2. If the bounded stopping times $t_n \to \infty$ satisfy

$$(6.3) \qquad E(X_{t_n} - \underline{X})^+ \to \phi^+(\Omega) , \qquad E|X_{t_n} - \underline{X}| \to \phi(\Omega) ,$$

then, for any sequence (A_n) *of sets with* $A_n \in \mathfrak{F}_{t_n}$ $(n \in N)$ *we have*

$$(6.4) \qquad E(X_{t_n} - \underline{X})1_{A_n} - \tilde{\phi}(A_n) \to 0\,, \qquad E|X_{t_n} - \underline{X}|1_{A_n} - \phi(A_n) \to 0\,.$$

In particular $E(X_{t_n} - \underline{X})1_A \to \tilde{\phi}(A)$ *for all* $A \in \mathcal{G}$.

Proof. We recall that (6.1) is assumed. First we show that the second condition in (6.3) implies the second assertion in (6.4). Suppose $\underline{\lim}(E|X_{t_n} - \underline{X}|1_{A_n} - \phi(A_n)) < 0$, then there exists $\varepsilon > 0$ and a subsequence for which $E|X_{t_n} - \underline{X}|1_{A_n} < \phi(A_n) - \varepsilon$. Without loss of generality we may assume that this inequality holds for all n. Let B_n be the complement of A_n. There exists $s_n > t_n$ satisfying $E|X_{s_n} - \underline{X}| < \phi(B_n) + \varepsilon/2$. But then, putting $\tau_n = t_n 1_{A_n} + s_n 1_{B_n}$ we have $E|X_{\tau_n} - \underline{X}| < \phi(A_n) + \phi(B_n) - \varepsilon/2 = \phi(\Omega) - \varepsilon/2$ which is impossible. If $\overline{\lim}(E|X_{t_n} - \underline{X}|1_{A_n} - \phi(A_n)) > 0$ then $\underline{\lim}\, E(|X_{t_n} - \underline{X}|1_{B_n} - \phi(B_n)) < 0$ which is again impossible. This establishes the second part of (6.4).

The rest of the Lemma will follow if we establish $E(X_{t_n} - \underline{X})^+1_{A_n} - \phi^+(A_n) \to 0$. Suppose, $E(X_{t_n} - \underline{X})^+1_{A_n} > \phi^+(A_n) + \varepsilon$ then we can construct a b.s.t. τ_n for which $E(X_{\tau_n} - \underline{X})^+ > \phi^+(\Omega) + \varepsilon/2$ and $E|X_{\tau_n} - \underline{X}| < E|X_{t_n} - \underline{X}|1_{A_n} + \phi(B_n) + 1/n$ which leads to a contradiction. The case $\underline{\lim}(E(X_{t_n} - \underline{X})^+1_{A_n} - \phi^+(A_n)) < 0$ is treated by considering the sequence (B_n). □

Lemma 6.3. If $Y \in C$ *there exists an increasing sequence of bounded stopping times* $t_n(Y)$ *such that* $X_{t_n(Y)} \to Y$ *a.s. and*

(6.5) $$E(X_{t_n(Y)} - Y)1_A \to \tilde{\phi}(A) ,$$

for every $A \in \mathcal{G}$.

Proof. Let $A_n \in \mathcal{G}$ be such that $P(A_n) > 1 - 1/n^2$ and $\phi(A_n) < 1/n$ (see Lemma 3.1). Let $B_n \in \mathcal{G}$ be a subset of A_n and $s_n(Y)$ be a b.s.t. such that $P(B_n) > 1 - 2/n^2$ and $E|X_{s_n(Y)} - Y|1_{B_n} < 1/n$ (see Lemma 4.1). Let (t_n) be a sequence of b.s.t. satisfying (6.3). We may assume that $(s_n(Y))$ and (t_n) are increasing and that $A_n, B_n \in \mathcal{F}_{s_n(Y)} \cap \mathcal{F}_{t_n}$. Put $t_n(Y) = s_n(Y)1_{B_n} + t_n 1_{C_n}$ where C_n is the complement of B_n . Then $X_{t_n(Y)} \to Y$ a.s. and

(6.6) $$E(X_{t_n(Y)} - Y)1_A = E(X_{s_n(Y)} - Y)1_{A\cap B_n} + E(Y - \underline{X})1_{A\cap C_n} + E(X_{t_n} - \underline{X})1_{A\cap C_n} .$$

The first two summands $\to 0$. Also $\tilde{\phi}(A\cap C_n) = \tilde{\phi}(A) - \tilde{\phi}(A\cap B_n)$, but $|\tilde{\phi}(A\cap B_n)| \le \phi(A\cap B_n) \le \phi(A_n)$. Thus $\phi(A\cap C_n) \to \tilde{\phi}(A)$ and hence, by the preceding Lemma, the last summand in (6.6) $\to \tilde{\phi}(A)$. □

An immediate consequence is

Theorem 6.1. If $\lim E|X_t| < \infty$ there exists for every $Y \in C$ an increasing sequence of bounded stopping times $t_n(Y)$ such that

(6.7) $$E\, X_{t_n(Y)} V \to E\, Y\, V + \sum_{i=1}^{m} \lambda_i \tilde{\phi}(A_i) ,$$

for every simple $\mathcal{G}$-*measurable random variable* $V = \sum_{i=1}^{m} \lambda_i 1_{A_i}$ $(A_i \in \mathcal{G},\ i=1,\dots m)$.

Corollary 6.1. *For all* $Y, Z \in C$ *and every simple* $\mathcal{G}$-*measurable random variable* V

$$\lim E(X_{t_i(Y)} - X_{t_j(Z)})V = E(Y - Z)V ,$$

as $i,j \to \infty$ *independently of one another*.

Remarks 6.1. There are, in general, many additive set functions $\tilde{\phi}$ for which Theorem 6.1 holds. E.g. if we replace $\overline{\lim}$ by $\underline{\lim}$ in (6.2) we obtain another, usually different , $\tilde{\phi}$ with the desired properties.

6.2. From the proof of Lemma 6.3 it is seen that the b.s.t. $t_n(Y)$ may be assumed to have simultaneously the properties described in Theorem 5.1 and Theorem 6.1.

6.3. Theorem 6.1 is stated for the case that C is not empty. If it is empty similar results hold for C_Z with $Z \in \overline{C}$. (See Corollary 4.2 and Remark 4.3).

7. Banach space valued random variables.

In this section we consider vector r.v. $\Omega \to S$ where S is a fixed Banach space (not necessarily over the reals).

Obviously nothing has to be changed in sections 2 and 3 beyond replacing the absolute value $|\cdot|$ by the norm $||\cdot||$ of S. Similarly $\overline{C}$ and C are defined by (4.1) with $||X_n - X||$ replacing $|X_n - X|$. C is a closed set in the relevant L_1 space (of r.v. Y with $E||Y|| < \infty$) and $\rho(Y,C)$ is defined as before.

At this stage there does occur an important difference. The fact that $\overline{C}(||X_n||)$ is not empty does not imply that $\overline{C}(X_n) \neq \emptyset$. Indeed, every infinite dimensional Banach space contains points e_n, $n \in \mathbb{N}$, of unit norm such that $||e_i - e_j|| \geq 1$ whenever $i \neq j$. Then $\overline{C}(e_n) = \emptyset$ whereas $C(||e_n||)$ is the constant 1. This fact affects some of the results in sections 4 and 5.

The approximation Lemma 4.1 remains valid (same proof).

Theorem 4.1 has to be modified, but the Corollary 4.1 is not affected. Thus we have

Theorem 7.1. For all integrable random variables Y we have

$$\phi(\Omega) \leq \underline{\lim}\, E||X_t - Y|| \leq \phi(\Omega) + \rho(Y,C) , \tag{7.1}$$

and if $\phi(\Omega) < \infty$ the first inequality becomes an equality when, and only when, $Y \in C$.

Proof. The first inequality (7.1) follows from $\phi(\cdot,(X_n)) = \phi(\cdot,(X_n - Y))$. The second inequality has to be proved only when $C \neq \emptyset$. If $X \in C$ then $\underline{\lim}\, E||X_t - Y|| \leq \underline{\lim}\, E(||X_t - X|| + ||X - Y||) = \underline{\lim}\, E||X_t - X|| + \rho(X,Y)$ but, by Lemma 4.1, $\underline{\lim}\, E||X_t - X|| \leq \phi(\Omega)$.

It remains to prove that if $\phi(\Omega) < \infty$ then equality on the left implies $Y \in C$. If $Y \notin C$ then there exists $\varepsilon > 0$ such that $E||X_t - Y||1_A > \varepsilon$ for every A with $P(A) > 1 - \varepsilon$ and large t. Let A be the set described in Lemma 3.1 with ε replaced by $\varepsilon/2$ and denote by B its complement. Then $\underline{\lim}\, E||X_t - Y|| \geq \underline{\lim}\, E||X_t - Y||1_B + \underline{\lim}\, E||X_t - Y||1_A \geq \phi(B) + \varepsilon > \phi(\Omega) + \varepsilon/2$. $\square$

Corollary 4.2 has to be modified in the same way as Theorem 4.1. Lemma 4.2 remains valid.

The results on simultaneous approximation (Theorems 5.1 and 5.2) remain unchanged.

The few reformulations of the results of sections 4 and 5 which were necessary in the general case are not needed when S is finite dimensional.

It is not difficult to prove that for finite dimensional Banach spaces S, if $Y \in \overline{C}(||X_n||)$ then there exists $Z \in \overline{C}(X_n)$ with $||Z|| = Y$.

Moreover, if S is a finite dimensional Banach space the results of section 6 also carry through.

Theorem 7.2. *Let* S *be a finite dimensional Banach space and* X_n *be* S*-valued random variables. If* $\underline{\lim}\, E||X_t|| < \infty$ *then* $C(X_n)$ *is not empty. Furthermore, there exists a finitely additive* S*-valued function* $\tilde{\phi}$ *with domain* $\mathcal{G}$ *and for every* $Y \in C$ *there exists a sequence of increasing bounded stopping times* $t_n(Y)$ *such that* (6.7) *holds for every scalar valued simple random variable* $V = \sum_{i=1}^{m} \lambda_i 1_{A_i}$ $(A_i \in \mathcal{G},\ i = 1,\dots,m)$.

The proof being similar to that of Theorem 6.1 we just show how to construct a suitable $\tilde{\phi}$. For brevity we do this for two-dimensional S over the reals. Let (e_1, e_2) be a basis of S and $X_n = X_n' e_1 + X_n'' e_2$ with real X_n', X_n''. Let $X' e_1 + X'' e_2 \in C$ be such that $||X'e_1 + X''e_2|| = \underline{\lim}\, E||X_t|| = \underline{X}$. Define, for $A \in \mathcal{G}$,

$$\phi_1(A) = \sup\{\overline{\lim}\, E|X_{t_n}' - X'|1_A ; \lim E||X_{t_n} - \underline{X}||1_A = \phi(A)\},$$

then let $\phi_2(A) = \sup\{\overline{\lim}|X_{t_n}'' - X''|;\ I, II\}$ where I is the condition used in defining ϕ_1 and II is the condition $\lim E|X_{t_n}' - X'|1_A = \phi_1(A)$. (If

S is stricly convex this step is not necessary since then ϕ_2 is determined by ϕ and ϕ_1.) Define $\phi_1^+(A) = \sup\{\overline{\lim}\, E(X'_{t_n} - X')^+; I, II\}$ and similarly $\phi_2^+(A)$. Let $\tilde{\phi}_i = 2\phi_i^+ - \phi_i$ $(i=1,2)$. $\tilde{\phi} = \tilde{\phi}_1 e_1 + \tilde{\phi}_2 e_2$ will have the required properties.

Remark 7.1. Theorem 7.2 fails in infinitely dimensional S, even if we add the requirement that $\mathcal{C} \neq \emptyset$. Indeed, let e_n be points in S with $||e_n|| = 1$ and $||e_i - e_j|| \geq 1$ for $i \neq j$. Let $\Omega = [0,1]$, P be the Lebesgue measure, $X_n = 2^n e_n 1_{[0,2^{-n}]}$ and $\mathfrak{F}_n = \sigma(X_1, \ldots, X_n)$. Then $\mathcal{C}$ is not empty but EX_t, $t \to \infty$, has no limit point in the norm topology. It is possible to obtain results similar to the above only if one either looks at weaker forms of convergence or imposes restrictions on the sequence X_n.

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